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Tugas Metode Matematika (D)

a. Tuliskan fungsi pembangkit untuk $P_n(x)$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

b. Dari fungsi pembangkit tersebut, buktikan relasi rekursi berikut ini.

$$(n+1) P_{n+1}(x) + n P_{n-1}(x) = (2n+1) x P_n(x)$$

Dari fungsi pembangkit pada poin a, yakni $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$

▶ Diferensialkan kedua ruas terhadap t

$$-\frac{1}{2}\left(1-2xt+t^2\right)^{-3/2} \cdot (-2x+2t) = \sum_{n=0}^{\infty} n \cdot t^{n-1} P_n(x)$$

$$-\frac{\frac{1}{2}(1-2xt+t^2)^{-1/2}}{(1-2xt+t^2)} \cdot (-2x+2t) = \sum_{n=0}^{\infty} n\, t^{n-1} P_n(x)$$

$$-\frac{1}{2}(-2x+2t) = (1-2xt+t^2) \sum_{n=0}^{\infty} n \cdot t^{n-1} P_n(x)$$

$$x-t = (1-2xt+t^2) \cdot \sum_{n=0}^{\infty} n \cdot t^{n-1} P_n(x)$$

$$(x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2) \cdot \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$x\sum_{n=0}^{\infty} t^n P_n(x) - t\sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) - 2x\sum_{n=0}^{\infty} n t^n P_n(x) + t\sum_{n=0}^{\infty} n t^n P_n(x)$$

$$x\sum_{n=0}^{\infty} t^n P_n(x) + 2x\sum_{n=0}^{\infty} n t^n P_n(x) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n(x) + \sum_{n=0}^{\infty} t^{n+1} P_n(x)$$

$$x P_n(x) + 2xn P_n(x) = (n+1) P_{n+1}(x) + (n-1) P_{n-1}(x) + P_{n-1}(x)$$

$$x P_n(x) + 2xn P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) - P_{n-1}(x) + P_{n-1}(x)$$

$$(n+1) P_{n+1}(x) + n P_{n-1}(x) = x P_n(x) + 2xn P_n(x)$$

$$(n+1) P_{n+1}(x) + n P_{n-1}(x) = (2n+1) x P_n(x) \quad \text{(TERBUKTI)}$$

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c. Dengan menggunakan relasi rekursif tersebut dapatkan $P_5(x)$ jika diketahui $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ dan $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$\to n = 4$ $\quad 5P_5(x) + 4P_3(x) = 9x \cdot P_4(x)$

$$5P_5(x) = 9x\left(\frac{1}{8}(35x^4 - 30x^2 + 3)\right) - 4 \cdot \frac{1}{2}(5x^3 - 3x)$$

$$= \frac{315x^5 - 270x^3 + 27x}{40} - \frac{(10x^3 - 6x)}{5}$$

$$P_5(x) = \frac{315x^5 - 270x^3 + 27x - 80x^3 + 48x}{40}$$
$$= \frac{315x^5 - 350x^3 + 75x}{40}$$
$$= \frac{1}{8}\left[63x^5 - 50x^3 + 15x\right]$$

3). Kembangkan fungsi $f(x) = \begin{cases} 2x+1 & ; \ 0 < x \le 1 \\ 0 & ; \ -1 \le x < 0 \end{cases}$ ke dalam deret $\sum_{n=0}^{\infty} a_n P_n(x)$ sampai 4 suku pertama.

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\, P_n(x)\, dx$$

▶ Untuk $n = 0$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\, P_0(x)\, dx$$
$$= \frac{1}{2}\left[\int_{-1}^{0} 0\, P_0(x)\, dx + \int_{0}^{1} (2x+1)\, dx\right]$$
$$= \frac{1}{2}\left[x^2 + x \Big|_0^1\right]$$
$$= \frac{1}{2} \cdot 2$$
$$= 1$$

▶ Untuk $n = 1$

$$a_1 = \frac{3}{2}\int_{-1}^{1} f(x)\, P_1(x)\, dx$$
$$= \frac{3}{2}\left[0 + \int_{0}^{1} (2x+1) . x\, dx\right]$$
$$= \frac{3}{2}\left[\int_{0}^{1} 2x^2 + x\, dx\right]$$
$$= \frac{3}{2}\left[\frac{2}{3}x^3 + \frac{1}{2}x^2 \Big|_0^1\right]$$
$$= \frac{3}{2}\left[\frac{2}{3} + \frac{1}{2}\right]$$
$$= \frac{3}{2}\left[\frac{4+3}{6}\right]$$
$$= \frac{3}{2} \cdot \frac{7}{6}$$
$$= \frac{7}{4}$$

▶ Untuk $n = 2$

$$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)\, P(x)\, dx$$
$$= \frac{5}{2}\left[\int_{-1}^{0} 0 . P_2(x)\, dx + \int_{0}^{1} (2x+1) . \frac{1}{2}(3x^2 - 1)\, dx\right]$$
$$= \frac{5}{4}\int_{0}^{1} 6x^3 - 2x + 3x^2 - 1\, dx$$
$$= \frac{5}{4}\left[\frac{3}{2}x^4 + x^3 - x^2 - x \Big|_0^1\right]$$
$$= \frac{5}{4}\left[\frac{3}{2} + 1 - 1 - 1\right]$$
$$= \frac{5}{4} \cdot \frac{1}{2}$$
$$= \frac{5}{8}$$

▶ Untuk $n = 3$

$$a_3 = \frac{7}{2}\int_{-1}^{1} f(x) . P_3(x)\, dx$$
$$= \frac{7}{2}\int_{0}^{1} (2x+1)\left(\frac{1}{2}(5x^3 - 3x)\right) dx$$
$$= \frac{7}{4}\int_{0}^{1} 10x^4 - 6x^2 + 5x^3 - 3x\, dx$$
$$= \frac{7}{4}\left[2x^5 - 2x^3 + \frac{5}{4}x^4 - \frac{3}{2}x^2 \Big|_0^1\right]$$
$$= \frac{7}{4}\left[2 - 2 + \frac{5}{4} - \frac{3}{2}\right]$$
$$= \frac{7}{4}\left[\frac{5-6}{4}\right]$$
$$= \frac{7}{4}\left[-\frac{1}{4}\right]$$
$$= -\frac{7}{16}$$

Sehingga, $f(x) = P_0(x) + \frac{7}{4}P_1(x) + \frac{5}{8}P_2(x) - \frac{7}{16}P_3(x)$

4). a. Nyatakan $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right)$

Gunakan rumus rekursi : $x J_{n+1}(x) = 2n J_n(x) - x J_{n-1}(x)$

↳ Kalikan rumus rekursi dengan $\frac{1}{x}$, diperoleh $J_{n+1}(x) = \frac{2}{x} n J_n(x) - J_{n-1}(x)$

$\# J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$\# J_{-n}(x) = (-1)^n J_n(x)$

$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

▶ Substitusikan $n = \frac{3}{2}$

$J_{5/2}(x) = \frac{2}{x} \cdot \frac{3}{2} J_{3/2}(x) - J_{1/2}(x)$ .... (1)

▶ Substitusikan $n = \frac{1}{2}$ untuk dapatkan nilai $J_{3/2}$

$J_{3/2}(x) = \frac{2}{x} \cdot \frac{1}{2} J_{1/2}(x) - J_{-1/2}(x)$

$= \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$ .... (2)

▶ Substitusi (2) ke (1)

$J_{5/2}(x) = \frac{3}{x}\left[\frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)\right] - J_{1/2}(x)$

$= \frac{3}{x^2} J_{1/2}(x) - \frac{3}{x} J_{-1/2}(x) - J_{1/2}(x)$

$= \frac{3}{x^2} \cdot \sqrt{\frac{2}{\pi x}} \sin x - \frac{3}{x}\sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$

$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right]$

$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3-x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right]$

b. Tunjukkan hasil integral $\int x^5 J_2(x)\,dx = x^5 J_3(x) + 2x^4 J_4(x) + C$

$\int x^5 J_2(x)\,dx =$